

# AN AP-STRUCTURE WITH FINSLERIAN FLAVOR: I

M.I.Wanas<sup>1</sup>

## Abstract

A geometric structure (FAP-structure), having both absolute parallelism and Finsler properties, is constructed. The building blocks of this structures are assumed to be functions of position and direction. A non-linear connection emerges naturally and is defined in terms of the building blocks of the structure. Two linear connections, one of Berwald type and the other of the Cartan type, are defined using the non-linear connection of the FAP. Both linear connections are non-symmetric and consequently admit torsion. A metric tensor is defined in terms of the building blocks of the structure. The condition for this metric to be a Finslerian one is obtained. Also, the condition for an FAP-space to be an AP-one is given.

## 1 Motivation

It is well known that the General Theory of Relativity (GR), the best known theory describing gravitational interactions, is suffering from many problems nowadays. For example, it cannot account for the flatness of the rotation curves of spiral galaxies [1]. Also, there are some problems concerning the interpretation of the accelerated expansion of the Universe [2] discovered using SN type Ia observations [3]. Since GR is successful in the domains of the solar system and binary pulsars, which have relatively small scales, some authors believe that its problems are scale problems (cf.[2]). Consequently, a different theory is required in order to deal with diverse ranges of scale.

The origin of problems of GR can be seen from a different point of view. The field equations of GR can be written as (cf.[4])

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (1)$$

where the L.H.S. is Einstein's tensor derived in the context of Riemannian geometry while the R.H.S. is a phenomenological tensor scaled by the constant  $\kappa$ . The later tensor is not a member of the geometric structure used. It describes the material-energy distribution in the system under consideration. In the case of empty space i.e. in the absence of material distribution, the field equations (1) will reduce to

$$R_{\mu\nu} = 0. \quad (2)$$

It is to be considered that all the success of GR comes from the use of (2), not (1), in application. Many of the problems of GR are connected to the use of (1) in applications. Einstein was the first who directed the attention to solve this problem [5]. This is manifested in his series of attempts to construct what has been called a " *Unified Field Theory*", in which tensors describing the material-energy distribution and fields are members of the

---

<sup>1</sup>Astronomy Department, Faculty of Science, Cairo University, Giza, Egypt.  
E-mail: wanas@frcu.eun.eg

geometric structure used. Some authors have attempted to follow his scheme for solving the problems of GR (cf. [6], [7]). The attempts in this direction concentrate on using different geometries of more wider structures. This is because the ten field variables ( $g_{\mu\nu}$ ) of the 4-dimensional Riemannian geometry are just sufficient to describe gravity alone. So, in order to accommodate more physical quantities and interactions, some authors have used extra dimensions (*e.g. Kaluza-Klein type Theories*). Others, have used different geometries (*e.g. Riemann-Cartan, Weyl-Cartan, Absolute Parallelism*). Absolute Parallelism (AP) geometry is more wider than the Riemannian one in the sense that, in 4-dimensions, it has six more degrees of freedom. Another advantage of the use of the AP-space <sup>2</sup>, in application, is that it has an associated Riemannian space. This facilitates a direct comparison of any theory, written in the AP-space, with GR. On the other hand, attention has been recently directed to Finsler geometry and its generalization to gain more insight on the infrastructure of physical phenomena (cf. [8], [9]).

The aim of the present work is to construct an AP-structure with Finslerian properties, in particular, an AP-structure with an associated Finsler space. This structure would facilitate:

1. The use of the advantages of both machineries of AP and Finsler geometries.
2. The gain of more information on the infrastructure of physical phenomena studied in AP-geometry.
3. Attribution of some physical meaning to objects of Finsler geometry, since AP-geometry has been used in many successful applications (for a brief review cf. [10]).
4. Solution of some of the problems of GR, mentioned above, if possible.

The paper is arranged as follows: In subsection 2.1 we give a brief review of some basic formulae of the AP-structure. In subsection 2.2. we give necessary formulae for Finsler structure. In Section 3 we construct an AP-structure using the machinery of Finsler geometry. Some concluding remarks are given in section 4.

## 2 Basic Formulae

In this section, we are going to give a brief review on basic formulae necessary for comparison with the results obtained in section 3.

### 2.1 Absolute Parallelism Space

An AP-space  $(M, \lambda)$  is an  $n$ -dimensional differentiable manifold  $(M)$  equipped with a set of  $n$ -linearly independent vectors  $\lambda_{\mu}$  (in what follows we are going to use Latin indices for vector numbers and Greek indices to indicate coordinate components. Each type of indices runs from 1 to  $n$ ). Since these vectors are linearly independent, the determinant of matrix  $(\lambda_{\mu})$  is non-vanishing. Consequently, we can define the contravariant components

---

<sup>2</sup>This geometric structure is known as "Parallelisable Manifold", relative to geometers.

of  $\lambda_{\mu}$  such that <sup>3</sup> ( for more details cf.[11])

$$\lambda_{\alpha} \lambda_i^{\beta} = \delta_{\alpha}^{\beta}, \quad (3)$$

$$\lambda_{\alpha} \lambda_j^{\alpha} = \delta_{ij}. \quad (4)$$

A linear connection is defined as

$$\Gamma_{\cdot\mu\nu}^{\alpha} \stackrel{\text{def}}{=} \lambda_i^{\alpha} \lambda_{\mu,\nu}, \quad (5)$$

where the comma (,) is used here to characterize ordinary partial differentiation relative to  $x^{\nu}$ . It is clear that the connection (5) is non-symmetric. So, one can define a torsion as

$$\Lambda_{\cdot\mu\nu}^{\alpha} \stackrel{\text{def}}{=} \Gamma_{\cdot\mu\nu}^{\alpha} - \Gamma_{\cdot\nu\mu}^{\alpha}, \quad (6)$$

which is a tensor of type (1,2), skew symmetric in its two lower indices. Now, using the vectors  $\lambda_i$ , the following second order symmetric tensors can be defined,

$$g_{\mu\nu} \stackrel{\text{def}}{=} \lambda_{\mu} \lambda_{\nu}, \quad (7)$$

$$g^{\alpha\beta} \stackrel{\text{def}}{=} \lambda_i^{\alpha} \lambda_i^{\beta}. \quad (8)$$

It is clear that these tensor, using (3), (4), satisfy the relation,

$$g_{\mu\alpha} g^{\mu\beta} = \delta_{\alpha}^{\beta}.$$

Using the above properties and knowing that  $g_{\mu\nu}$  is non degenerate, then (7) can be used as a metric tensor of a Riemannian space associated with any AP-structure. Also, the tensors (7) and (8) can be used to lower or raise tensor indices. Using (7) and (8) we can define as usual a linear symmetric connection, which is Christoffel symbol of the second kind

$$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \stackrel{\text{def}}{=} \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \quad (9)$$

We can use the connections (5) and (9) to perform covariant differentiation as follows,

$$A_{\mu+|\nu} \stackrel{\text{def}}{=} A_{\mu,\nu} - A_{\alpha} \Gamma_{\cdot\mu\nu}^{\alpha}, \quad (10)$$

$$A_{\mu;\nu} = A_{\mu,\nu} - A_{\alpha} \left\{ \begin{matrix} \alpha \\ \nu\mu \end{matrix} \right\}, \quad (11)$$

where  $A_{\mu}$  is an arbitrary covariant vector.

Applying the types of differentiation given by (10) and (11) to the vectors  $\lambda_i$  and the tensor (7) we get the following results,

$$\lambda_{i+|\nu}^{\mu} = 0, \quad (12)$$

---

<sup>3</sup>Latin indices (vector numbers) are always written in a lower position neither covariant nor contravariant. Summation convention is applied over repeated indices whatever their positions.

$$g_{\mu\nu|+}^{\sigma} = 0, \quad (13)$$

$$g_{\mu\nu;\sigma} = 0. \quad (14)$$

Equations (12) indicates absolute parallelism while (13) and (14) are metricity conditions. The tensors  $g^{\mu\nu}$  and  $g_{\mu\nu}$  can be used to raise and lower coordinate indices under the above covariant differentiation signs.

A third order tensor ,*contortion*, can be defined as

$$\gamma_{\cdot\mu\nu}^{\alpha} \stackrel{\text{def}}{=} \lambda_i^{\alpha} \lambda_{\mu;\nu}^i, \quad (15)$$

which is non-symmetric with respect to its lower two indices. It can be shown that

$$\Lambda_{\cdot\mu\nu}^{\alpha} = \gamma_{\cdot\mu\nu}^{\alpha} - \gamma_{\cdot\nu\mu}^{\alpha}, \quad (16)$$

$$C_{\mu} = \Lambda_{\cdot\mu\alpha}^{\alpha} = \gamma_{\cdot\mu\alpha}^{\alpha}, \quad (17)$$

where  $C_{\mu}$  is a covariant vector, *the basic vector*. The curvature tensor corresponding to (5) vanishes identically, due to (12), but the curvature tensors corresponding to its dual  $\tilde{\Gamma}_{\mu\nu}^{\alpha} (= \Gamma_{\nu\mu}^{\alpha})$  and that corresponding to its symmetric part  $\Gamma_{(\mu\nu)}^{\alpha}$  do not vanish [11].

## 2.2 Finsler Structure

A Finsler space  $(M, F)$  is an n-dimensional differentiable manifold  $M$  equipped with a scalar  $F(x, y)$  function of  $x(t), y(t) (= \frac{dx}{dt})$  where  $t$  is an invariant parameter (for more details cf.[12]),  $(x, y)$  are the coordinates on the tangent bundle  $TM$ . Now, the scalar  $F(x, y)$  is assumed to satisfy the following properties:

- 1-  $F(x, y)$  is  $C^{\infty}$  on  $\tau M (= TM \setminus \{0\})$ .  $F$  is called the fundamental function.
- 2- The function  $F(x, y)$  is positively homogenous of degree one in  $y$ , abbreviated as  $(P - h(1))$ .
- 3-  $y \in \tau M$  and transforms as,

$$\bar{y}^{\alpha} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}} y^{\beta}. \quad (18)$$

- 4- The tensor whose component defined by,

$$g_{\alpha\beta} \stackrel{\text{def}}{=} \frac{\partial^2 E}{\partial y^{\alpha} \partial y^{\beta}} = E :_{\alpha\beta}, \quad (19)$$

is positive definite (non-degenerate). This tensor defines the metric of Finsler space, where  $E$  is the energy of this space and the colon  $(:)$  is used to characterize differentiation with respect to  $y$ . This tensor is symmetric and is  $P - h(0)$ . The function  $E$  is defined by,

$$E \stackrel{\text{def}}{=} \frac{1}{2} F^2. \quad (20)$$

Consequently,

$$F^2 \stackrel{\text{def}}{=} g_{\alpha\beta} y^{\alpha} y^{\beta}. \quad (21)$$

A tensor  $C_{\alpha\beta\gamma}$  defined by,

$$C_{\alpha\beta\gamma} \stackrel{\text{def}}{=} g_{\alpha\beta;\gamma} = E_{:\alpha\beta\gamma}, \quad (22)$$

has the following properties:

1. It is symmetric with respect to all indices.
2. Tensor of type (0,3).
3.  $P - h(-1)$ .

Using Euler's Theorem we get,

$$C_{\alpha\beta\gamma} y^\alpha = C_{\alpha\beta\gamma} y^\beta = C_{\alpha\beta\gamma} y^\gamma = 0. \quad (23)$$

Since the metric tensor is assumed non-degenerate its conjugate can be defined such that,

$$g^{\alpha\beta} g_{\alpha\gamma} = \delta_\gamma^\beta. \quad (24)$$

Consequently, these tensors can be used to perform the operations of raising and lowering indices. The following theorem can be easily proved.

**Theorem:**

A necessary and sufficient condition for a Finsler space to be a Riemannian one is that the tensor  $C_{\alpha\beta\gamma}$  vanishes.

**A Non-Linear Connection**

Using the metric tensor and its conjugate, one can define the object,

$$G_{\cdot\beta\gamma}^\alpha \stackrel{\text{def}}{=} \frac{1}{2} g^{\alpha\sigma} (g_{\beta\sigma;\gamma} + g_{\sigma\gamma;\beta} - g_{\beta\gamma;\sigma}). \quad (25)$$

Since  $g_{\mu\nu}$  is a function of  $(x, y)$ , then  $G_{\cdot\beta\gamma}^\alpha$  is neither a tensor nor a connection. It is  $P - h(0)$ . Consider the quantity,

$$G^\alpha \stackrel{\text{def}}{=} G_{\cdot\beta\gamma}^\alpha (x, y) y^\beta y^\gamma. \quad (26)$$

It can be shown that (26) defines a contravariant vector, called **spray**, and it is  $P - h(2)$ . The quantity

$$G_{\cdot\sigma}^\alpha \stackrel{\text{def}}{=} G_{\cdot\sigma}^\alpha = \frac{\partial G^\alpha}{\partial y^\sigma}, \quad (27)$$

can be shown to transform according to the law,

$$\bar{G}_{\cdot\beta}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial \bar{x}^\beta} G_{\cdot\nu}^\mu + \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \bar{y}^\gamma. \quad (28)$$

So, this quantity represents the components of a non-linear connection. It is  $P - h(1)$ . It can be used to define the differential operator.

$$\delta_\mu \stackrel{\text{def}}{=} \partial_\mu - G_{\cdot\mu}^\nu \frac{\partial}{\partial y^\nu} \quad (29)$$

### Cartan Linear Connection

This connection is defined as

$$*\Gamma_{\cdot\beta\gamma}^\alpha \stackrel{\text{def}}{=} \frac{1}{2}g^{\sigma\alpha}(\delta_\gamma g_{\beta\sigma} + \delta_\beta g_{\sigma\gamma} - \delta_\sigma g_{\beta\gamma}) \quad (30)$$

which can be shown to transform according to the linear connection transformation,

$$*\bar{\Gamma}_{\cdot\beta\gamma}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial \bar{x}^\beta} \frac{\partial x^\sigma}{\partial \bar{x}^\gamma} * \Gamma_{\cdot\nu\sigma}^\mu + \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \quad (31)$$

This connection is a metric one.

### Berwald Linear Connection

This linear connection is defined by,

$$*G_{\cdot\beta\gamma}^\alpha \stackrel{\text{def}}{=} \frac{G_{\cdot\gamma}^\alpha}{\partial y^\beta} = G_{\gamma;\beta}^\alpha = G_{\cdot\beta\gamma}^\alpha, \quad (32)$$

and can be shown to transform according to (31). This connection is non-metric. It has no torsion since it is symmetric w.r.t. its lower two indices.

## 3 AP-Space With Finslerian Properties

**definition:** An Absolute Parallelism space with Finslerian properties  $(M, L_i)$  is an  $n$ -dimensional differentiable manifold  $M$  equipped with a set of  $n$ -Lagrangian functions  $L_i = L_i(x, y)$  having the following properties:

1.  $L_i(x, y)$  is  $C^\infty$  on  $\tau M (= TM \setminus \{0\})$ .
2.  $L_i(x, y) > 0, y \in \tau M, y = \dot{x}$ .
3.  $L_i(x, y)$  is positively homogenous of degree one, i.e.  $P - h(1)$ .
4. The vectors defined by

$$\lambda_{\mu}^i(x, y) \stackrel{\text{def}}{=} \frac{\partial L_i}{\partial y^\mu} \quad (33)$$

are assumed to be linearly independent. We are going to abbreviate this space by FAP-space. The set of Lagrangian functions  $L_i (i = 1, 2, 3, \dots, n)$  is called the fundamental set. The set of functions  $\lambda_{\mu}^i(x, y)$  are  $P - h(0)$  and are the building blocks of the FAP-space. It can be easily shown that  $\lambda_{\mu}^i$  transforms as components of covariant vectors under the group of general coordinate transformations.

Now, using Euler's theorem and definition (33), we get

$$L_i = \lambda_{\mu}^i y^\mu. \quad (34)$$

Let us define the functions,

$$C_{\mu\alpha}^i \stackrel{\text{def}}{=} \frac{\partial \lambda_{\mu}^i}{\partial y^\alpha} = L_{i;\mu\alpha}. \quad (35)$$

Again using Euler's theorem, we find that

$$C_{\mu\alpha} y^\mu = C_{\mu\alpha} y^\alpha = 0 \quad (36)$$

Now, the object  $C_{\mu\alpha}$  ( $\stackrel{\text{def}}{=} \frac{\partial \lambda_\mu}{\partial y^\alpha} = \lambda_{\mu;\alpha}$ ) has the following properties:

- 1-  $C_{\mu\alpha}$  is a tensor of type  $(0, 2)$ .
- 2-  $C_{\mu\alpha}$  is  $P - h(-1)$ .
- 3-  $C_{\mu\alpha}$  is symmetric with respect to its two tensor indices, as clear from (35).

**Theorem I:** "A necessary and sufficient condition for an FAP-space to be an AP-space is that  $C_{\alpha\beta}$  vanishes identically".

This can be easily proved. As stated above,  $\lambda_\alpha$  are totally independent, then the matrix  $(\lambda_\alpha)$  is non-degenerate. Consequently, we can define  $\lambda^\mu$  such that:

$$\lambda^\mu_{\phantom{\mu}i} \lambda_\nu^i = \delta^\mu_\nu, \quad (37),$$

$$\lambda^\mu_{\phantom{\mu}i} \lambda_\mu^j = \delta_{ij}, \quad (38).$$

**Definitions:** let

$$C_{\phantom{\mu}i}^\mu{}_{\phantom{\mu}\nu} \stackrel{\text{def}}{=} \frac{\partial \lambda^\mu_i}{\partial y^\nu} = \lambda^\mu_{\phantom{\mu}i;\nu} \quad (39)$$

$$\hat{C}_{\phantom{\mu}i}^\mu{}_{\phantom{\mu}\beta\gamma} \stackrel{\text{def}}{=} \lambda^\mu_{\phantom{\mu}i} C_{\phantom{\mu}\beta\gamma}, \quad (40)$$

where  $C_{\beta\gamma}$  is the tensor given by (35). Multiplying both sides of (40) by  $\lambda_\mu^j$ , then we get the following relation between (35) and (40),

$$\lambda_\mu^j \hat{C}_{\phantom{\mu}i}^\mu{}_{\phantom{\mu}\beta\gamma} = C_{\phantom{\mu}\beta\gamma}^j. \quad (41)$$

These tensors will be used later.

### A Non-Linear Connection

Using the transformation of the operator  $\frac{\partial}{\partial \bar{x}}$  [12], as a coordinate vector fields on TM, we can write

$$\begin{aligned} \frac{\partial \bar{\lambda}_\mu}{\partial \bar{x}^\alpha} &= \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \frac{\partial \bar{\lambda}_\mu}{\partial x^\beta} + \frac{\partial^2 x^\beta}{\partial \bar{x}^\alpha \partial \bar{x}^\sigma} \bar{y}^\sigma \frac{\partial \bar{\lambda}_\mu}{\partial y^\beta} \\ &= \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \frac{\partial}{\partial x^\beta} \left( \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \lambda_\gamma \right) + \frac{\partial^2 x^\beta}{\partial \bar{x}^\alpha \partial \bar{x}^\sigma} \bar{y}^\sigma \frac{\partial}{\partial y^\beta} \left( \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \lambda_\gamma \right) \\ &= \frac{\partial^2 x^\gamma}{\partial \bar{x}^\alpha \partial \bar{x}^\mu} \lambda_\gamma + \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \frac{\partial \lambda_\gamma}{\partial x^\beta} + \frac{\partial^2 x^\beta}{\partial \bar{x}^\alpha \partial \bar{x}^\sigma} \bar{y}^\sigma \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \lambda_{\gamma;\beta}, \\ \frac{\partial \bar{\lambda}_\mu}{\partial \bar{x}^\alpha} &= \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \lambda_{\gamma;\beta} + \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\alpha} \lambda_\gamma + \frac{\partial^2 x^\beta}{\partial \bar{x}^\sigma \partial \bar{x}^\alpha} \bar{y}^\sigma \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \lambda_{\gamma;\beta}. \end{aligned} \quad (42)$$

Multiplying both sides by  $\bar{\lambda}_i^\nu \bar{y}^\mu$  we get after some reductions,

$$(\bar{y}^\mu \bar{\lambda}_i^\nu \frac{\partial \bar{\lambda}_\mu^-}{\partial \bar{x}^\alpha}) = \frac{\partial \bar{x}^\nu}{\partial x^\epsilon} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} (y^\gamma \lambda_i^\epsilon \frac{\partial \lambda_\gamma}{\partial x^\beta}) + \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\gamma} \bar{y}^\mu + \frac{\partial^2 x^\beta}{\partial \bar{x}^\alpha \partial \bar{x}^\sigma} \frac{\partial \bar{x}^\nu}{\partial x^\epsilon} \bar{y}^\sigma y^\gamma \hat{C}_{\gamma\beta}^\epsilon. \quad (43)$$

Now, using definition (40) and the properties of  $C_{\gamma\beta}$ , given above, we have,

$$y^\gamma \hat{C}_{\gamma\beta}^\epsilon = y^\gamma \lambda_i^\epsilon C_{\gamma\beta} = 0.$$

This will cause the vanishing of the last term of (43). Then the quantities between brackets in this equation transform as

$$\bar{N}_{\cdot\alpha}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\epsilon} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} N_{\cdot\beta}^\epsilon + \frac{\partial \bar{x}^\nu}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\alpha \partial \bar{x}^\mu} \bar{y}^\mu. \quad (44)$$

where,

$$N_{\cdot\alpha}^\nu \stackrel{\text{def}}{=} y^\mu \lambda_i^\nu \lambda_{\mu,\alpha}. \quad (45)$$

Comparing (44) with (28), one can conclude that  $N_{\cdot\alpha}^\nu$ , defined by (45), represents the components of a non-linear connection defined in FAP-space.

### Berwald-Like Linear Connection

Differentiating both sides of (44) w.r.t.  $\bar{y}^\sigma$ , we get (note that  $\frac{\partial}{\partial y}(\frac{\partial x}{\partial \bar{x}}) = 0$ )

$$\begin{aligned} \bar{N}_{\cdot\alpha;\sigma}^\nu &= \frac{\partial \bar{x}^\nu}{\partial x^\epsilon} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \frac{\partial x^\gamma}{\partial \bar{x}^\sigma} N_{\cdot\beta;\gamma}^\epsilon + \frac{\partial \bar{x}^\nu}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\alpha \partial \bar{x}^\mu} \delta_\sigma^\mu \\ &= \frac{\partial \bar{x}^\nu}{\partial x^\epsilon} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \frac{\partial x^\gamma}{\partial \bar{x}^\sigma} N_{\cdot\beta;\gamma}^\epsilon + \frac{\partial \bar{x}^\nu}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\alpha \partial \bar{x}^\sigma}. \end{aligned} \quad (46)$$

Comparing (46) with (31), one can conclude that the quantity  $\bar{N}_{\cdot\alpha;\sigma}^\nu$  transforms as a linear connection and will be denoted by

$$B_{\cdot\alpha\sigma}^\nu \stackrel{\text{def}}{=} N_{\cdot\alpha;\sigma}^\nu = \frac{\partial}{\partial y^\sigma} (y^\mu \lambda_i^\nu \lambda_{\mu,\alpha}). \quad (47)$$

It is clear that  $B_{\cdot\alpha\mu}^\nu$  is non-symmetric in its lower two indices. So, it has a torsion.

### Cartan-Like Linear Connection

The non-linear connection defined by (45) can be used to define the operator  $\delta$ , similar to (29),

$$\delta_\mu \stackrel{\text{def}}{=} \partial_\mu - N_{\cdot\mu}^\alpha \frac{\partial}{\partial y^\alpha}. \quad (48)$$

So, if  $A_\alpha$  is an arbitrary vector, then we can define the derivative,

$$\delta_\beta A_\alpha = A_\alpha \cdot_\beta \stackrel{\text{def}}{=} A_{\alpha,\beta} - N_{\cdot\beta}^\gamma A_{\alpha;\gamma}. \quad (49)$$



Using the operator (48), we can define the object,

$$\Gamma_{\alpha\beta}^\mu \stackrel{\text{def}}{=} \lambda_i^\mu \lambda_{\alpha\hat{\beta}}. \quad (50)$$

Such quantities can be shown to transform according to (31). So, the set of quantities, given by (50) represents the components of a linear connection, different from (47), and of Cartan type. It is non-symmetric w.r.t. its lower two indices. Consequently, it admits a torsion.

### Covariant V- and H- Derivatives:

Let us define the following vertical (V-) covariant derivative

$$A_\mu|_\nu \stackrel{\text{def}}{=} A_{\mu:\nu} + A_\alpha \hat{C}_{\mu\nu}^\alpha, \quad (51)$$

where,

$$A_{\mu:\nu} = \frac{\partial A_\mu}{\partial y^\nu} \quad (52)$$

and

$$\hat{C}_{\mu\nu}^\alpha = \lambda_i^\alpha \frac{\partial^2 L_i}{\partial y^\mu \partial y^\nu} \quad (53)$$

Now, for the building blocks of the FAP-space, using (51) we get

$$\lambda_i|_\beta \equiv 0. \quad (54)$$

This implies that  $\lambda_i|_\beta$  are parallel displaced along a certain path characterized by  $\hat{C}_{\alpha\beta}^\gamma$  (absolute parallelism). Consequently, using (37), we can show that,

$$\lambda_i^\mu|_\nu \equiv 0. \quad (55)$$

We can also define the H-covariant derivative as

$$A_{\alpha\parallel\beta} = A_{\alpha\hat{\beta}} - A_\mu \Gamma_{\alpha\beta}^\mu, \quad (56)$$

where  $A_{\alpha\hat{\beta}}$  is given by (49) and  $\Gamma_{\alpha\beta}^\mu$  is the linear connection defined by (50). Consequently, we get

$$\lambda_{\alpha\parallel\beta} \equiv 0. \quad (57)$$

Similarly, we can write

$$\lambda_i^\beta|_\gamma \equiv 0. \quad (58)$$

The relation (57) shows that  $\lambda_{\alpha\parallel\beta}$  are parallel displaced along a certain path characterized by the linear connection (50).

### Introduction of a Metric

So far, all the above calculations have been performed without using a metric. This is achieved by using the building blocks of the FAP-space. But due to the importance of the metric tensor in physical applications and in order to extend the geometric structure, it is preferable to introduce a metric in this stage. For this reason, consider the following quantities: Let us define the second order symmetric tensors

$$g_{\mu\nu} \stackrel{\text{def}}{=} \lambda_{\mu} \lambda_{\nu}, \quad (59)$$

$$g^{\alpha\beta} \stackrel{\text{def}}{=} \lambda^{\alpha} \lambda^{\beta}. \quad (60)$$

Consequently, using the relations (37) and (38) on the building blocks of the FAP-space, we can write

$$g^{\alpha\mu} g_{\beta\mu} = \delta_{\beta}^{\alpha}.$$

Consider now the quantity,

$$\begin{aligned} C_{\alpha\beta\gamma} &\stackrel{\text{def}}{=} \frac{1}{2} \frac{\partial g_{\beta\gamma}}{\partial y^{\alpha}} \\ &= \frac{1}{2} \frac{\partial}{\partial y^{\alpha}} (\lambda_{\beta} \lambda_{\gamma}) \\ &= \frac{1}{2} (\lambda_{\beta} \lambda_{\gamma:\alpha} + \lambda_{\beta:\alpha} \lambda_{\gamma}) \\ &= \frac{1}{2} (C_{\gamma\alpha} \lambda_{\beta} + C_{\beta\alpha} \lambda_{\gamma}). \end{aligned} \quad (61)$$

If we use the definition,

$$\hat{C}_{\alpha\beta\gamma} \stackrel{\text{def}}{=} C_{\alpha\beta} \lambda_{\gamma}$$

then we can write,

$$\hat{C}_{\alpha\beta\gamma} = \hat{C}_{\alpha(\beta\gamma)} + \hat{C}_{\alpha[\beta\gamma]}, \quad (62)$$

where,

$$\hat{C}_{\alpha(\beta\gamma)} \stackrel{\text{def}}{=} \frac{1}{2} (\hat{C}_{\alpha\beta\gamma} + \hat{C}_{\alpha\gamma\beta}), \quad (63)$$

$$\hat{C}_{\alpha[\beta\gamma]} \stackrel{\text{def}}{=} \frac{1}{2} (\hat{C}_{\alpha\beta\gamma} - \hat{C}_{\alpha\gamma\beta}). \quad (64)$$

From (61) and (63) we can write  $C_{\alpha\beta\gamma} = \hat{C}_{\alpha(\beta\gamma)}$ . The tensor  $C_{\alpha\beta\gamma}$  has the following properties:

- 1- It is a tensor of type (0,3), symmetric w.r.t. all indices.
- 2- It is  $P - h(-1)$ .
- 3-  $C_{\alpha\beta\gamma} y^{\alpha} = C_{\alpha\beta\gamma} y^{\beta} = C_{\alpha\beta\gamma} y^{\gamma} = 0$  (using Euler's theorem). It is to be considered that the tensor (61) has the same properties of the tensor given by (22), although it is defined in terms of the building blocks of the FAP-space. For this reason the two tensors are referred to using the same symbol.

Let us define the energy of FAP-space,

$$E \stackrel{\text{def}}{=} \frac{1}{2} L^2 = \frac{1}{2} \sum_i L_i L_i = \frac{1}{2} \lambda_{\alpha} \lambda_{\beta} y^{\alpha} y^{\beta}, \quad (65)$$

such that  $E$  is  $P - h(2)$ , then

$$\begin{aligned} \frac{\partial E}{\partial y^{\sigma}} &= \frac{1}{2} \lambda_{\alpha} \lambda_{\beta} \delta_{\sigma}^{\alpha} y^{\beta} + \frac{1}{2} \lambda_{\alpha} \lambda_{\beta} \delta_{\sigma}^{\beta} y^{\alpha} + \frac{1}{2} (\lambda_{\alpha} \lambda_{\beta:\sigma} + \lambda_{\beta} \lambda_{\alpha:\sigma}) y^{\alpha} y^{\beta} \\ &= \lambda_{\sigma} \lambda_{\beta} y^{\beta} + C_{\alpha\beta\sigma} y^{\alpha} y^{\beta} \\ &= \lambda_{\sigma} \lambda_{\beta} y^{\beta} \quad (\text{using (23)}). \\ \frac{\partial^2 E}{\partial y^{\sigma} \partial y^{\epsilon}} &= \lambda_{\sigma:\epsilon} \lambda_{\beta} y^{\beta} + \lambda_{\beta:\epsilon} \lambda_{\sigma} y^{\beta} + \lambda_{\sigma} \lambda_{\epsilon} \\ &= \lambda_{\sigma} \lambda_{\epsilon} + 2C_{\epsilon\beta\sigma} y^{\beta} \\ &= g_{\sigma\epsilon}. \quad (\text{using (23)}). \end{aligned} \quad (66)$$

Now, the object  $g_{\mu\nu}$  has the following properties:

- 1- It is a symmetric tensor of type (0,2).
  - 2- It is non-degenerate since  $\lambda$  is non-degenerate.
  - 3- It is  $P - h(0)$ , positively homogenous of degree 0 in  $y$ .
  - 4- It is derivable from the energy of the FAP-space,  $E = \frac{1}{2} \sum_i L_i L_i$ , as given by (66).
- So, it can be used as a metric tensor of a space associated with the FAP-space.

### Metricity Conditions

Equation (54) implies that

$$g_{\alpha\beta}|_{\gamma} \equiv 0. \quad (67)$$

But we have

$$g^{\mu\beta} g_{\mu\alpha} = \delta_{\alpha}^{\beta}$$

Consequently, we get

$$g^{\epsilon\beta}|_{\sigma} \equiv 0 \quad (68)$$

From (67) and (68) a metricity condition is automatically satisfied. This implies that the operator of raising and lowering indices commutes with the V-covariant differential operator. In other words, one can use  $g^{\mu\nu}$  and  $g_{\alpha\beta}$  for raising and lowering tensor indices, under V-covariant operator sign, respectively.

Also, as a direct consequence of (58) and (59) we can write

$$g_{\mu\nu}|_{\sigma} \equiv 0, \quad (69)$$

$$g^{\mu\nu}|_{\sigma} \equiv 0, \quad (70)$$

which shows that (50) defines a metric linear connection. This shows that the tensor  $g_{\alpha\beta}$  and its conjugate can be used to lower and raise tensor indices under the H-covariant

differentiation sign, respectively.

**A Missing Condition:** Although we have shown that the metric can be derived from the energy of the FAP-space, as given by (66), it implies a certain condition as shown below.

**Theorem II:** In case of  $\hat{C}_{\alpha[\beta\gamma]=0}$  the FAP-space has an associated Finsler space whose metric is given by

$$g_{\alpha\beta} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^\alpha \partial y^\beta}$$

**proof**

$$\begin{aligned} g_{\mu\nu} &= \frac{1}{2} \frac{\partial^2 L^2}{\partial y^\mu \partial y^\nu} = \frac{\partial}{\partial y^\mu} \left( \frac{\partial L_i}{\partial y^\nu} L_i \right), \\ &= \frac{\partial L_i}{\partial y^\nu} \frac{\partial L_i}{\partial y^\mu} + L_i \frac{\partial^2 L_i}{\partial y^\mu \partial y^\nu} \\ &= \lambda_{\mu i} \lambda_{\nu i} + \lambda_{\alpha i} y^\alpha C_{\mu\nu i} \\ &= \lambda_{\mu i} \lambda_{\nu i} + y^\alpha \hat{C}_{\mu\nu\alpha} \\ &= \lambda_{\mu i} \lambda_{\nu i} + y^\alpha (\hat{C}_{\mu(\nu\alpha)} + \hat{C}_{\mu[\nu\alpha]}) \\ &= \lambda_{\mu i} \lambda_{\nu i} + y^\alpha C_{\mu\nu\alpha} + y^\alpha \hat{C}_{\mu[\nu\alpha]} \\ &= \lambda_{\mu i} \lambda_{\nu i}. \end{aligned} \tag{71}$$

Therefore the above theorem is proved.

## 4 Concluding Remarks

1- In the present article, the bases of a parallelisable structure with Finslerian properties, have been suggested. It possesses both advantages of the conventional AP-geometry and Finser geometry.

2- Theorem I gives the necessary and sufficient condition for an FAP-structure to be an AP-one. The vanishing of the tensor (35) is a mathematical expression for this condition. This condition would be very useful in applications. For a physical application written in the FAP-space, it is easy to get its AP-picture, and consequently its Riemannian one, just by using this condition. This would help in attributing some physical meaning to Finslerian geometric objects.

3- All geometric quantities, of the FAP-structure given in the present work, are defined in terms of the building blocks of this structure. It is worth of mention that the linear connection given by (50) is similar to that defined in reference [14], but in the present work we use the non-linear connection (45) that is defined in terms of the building blocks of the FAP-structure.

- 4- The condition for the metric tensor (59) to be a Finslerian metric is given by theorem II. This facilitates comparison between any physical application written in the FAP-structure and its Finslerian picture, if any.
- 5- In the structure developed, we meant by the torsion the anti-symmetric part of any linear connection.
- 6- In the present part of the work, we focused on the building blocks, connections, covariant derivatives and metric. The study of curvatures, the W-tensor [13], torsion and identities will be given in the next part of this work.

## Acknowledgments

The author would like to thank Professor N.L. Youssef, Dr. S. Abed and Mr. A. Sid-Ahmed for many discussions and comments.

## References

- [1] Y. Sofue and V. Rubin (2001) Ann.Rev.Astron.Astrophys. **39**, 137-174; astro-ph/0010594.
- [2] P.D. Mannheim (2006) Prog.Part.Nucl.Phys. **56**, 340-445; astro-ph/0505266.
- [3] J.L. Tonry, P.B. Schmidt et al. (2003) Astrophys.J. **594**, 1-24; astro-ph/0305008.
- [4] S.Weinberg (1972) *"Gravitation and Cosmology"*, John Wiley & Sons.
- [5] A. Einstein (1955) *"The Meaning of Relativity"*, App. II, Princeton, 5th ed.
- [6] G. Shipov (1998) *"A Theory of Physical Vacuum"*, Moscow.
- [7] F.I. Mikhail and M.I. Wanas (1977) Proc.Roy.Soc.Lond. **A356**, 471-481.
- [8] R. Miron and M. Anastasiei (1997) *"Vector Bundles and Lagrange Spaces with Applications to Relativity"*, Balakan Press.
- [9] G.S. Assanov and S.F. Ponomarenko (1989) *"Finsler Bundles Over Space-Time: Associated Gauge Fields and Connections"*, Nauka.
- [10] M.I. Wanas (2000) Turk.J.Phys. **24**, 473-488; gr-qc/0010099.
- [11] M.I. Wanas (2001) Cercet.Stiin.Ser.Mat. **10**, 297-309; gr-qc/0209050.
- [12] D. Bao, S.S. Chern and Z. Shen (2000) *"An Introduction to Riemann-Finsler Geometry"*, Springer.
- [13] N.L. Youssef and A.M. Sid-Ahmed (2006) Rep.Math.Phys. **60**, 39-53; gr-qc/0604111.
- [14] M.I. Wanas, N.L. Youssef and A.M. Sid-Ahmed (2007) arxiv 0704.2001.